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Hydromagnetic screw dynamo

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We solve the problem of magnetic field generation by a laminar flow of conducting fluid with helical (screw-like) streamlines for large magnetic Reynolds numbers, R_m. Asymptotic solutions are obtained with help of the singular perturbation theory. The generated field concentrates within cylindrical layers whose position, the magnetic field configuration and the growth rate are determined by the distribution of the angular, Ω, and longitudinal, V_z, velocities along the radius. The growth rate is proportional to $R_m^{-\frac{1}{3}}$. When Ω and V_z are identically distributed along the radius, the asymptotic forms are of the WKB type; for different distributions, singular-layer asymptotics of the Prandtl type arise. The solutions are qualitatively different from those obtained for solid-body screw motion. The generation threshold strongly depends on the velocity profiles.

1. Introduction

A laminar flow with helical (screw-like) streamlines is known to be one of the simplest flows of a conducting fluid that is capable of magnetic field generation. This property was discovered by Lortz (1968) and Ponomarenko (1973), and further investigated by Gailitis & Freiberg (1976, 1980) and recently by Roberts (1987) for a model of a solid-body conductive cylinder in axial helical motion. The generation properties of a helical Couette–Poiseuille flow have been numerically investigated by Solovyev (1985, 1987).

The kinematic screw dynamo considered by Ponomarenko was classified as a slow dynamo: the growth rate of any magnetic mode tends to zero when the magnetic Reynolds number increases (see Zeldovich, Ruzmaikin & Sokoloff 1983, §4.3). This property is due to the fact that magnetic diffusivity plays a key role in the generation process, producing the radial magnetic field component at the expense of the azimuthal one in a cylindrically symmetric laminar flow. The differential rotation (concentrated at the cylinder surface in the case of solid-body motion) produces an azimuthal field from the radial one while the longitudinal shear, which draws apart oppositely directed magnetic lines, opposes a destructive decrease in the radial scale of the generated non-axisymmetric magnetic field. However, Gilbert (1988) notes that even though the growth rate of any mode decreases with the magnetic Reynolds number R_m, at any given R_m the growth rate is maximal for a certain (small-scale) mode with azimuthal and longitudinal scales proportional to $R_m^{\frac{1}{3}}$. Moreover, the value of this maximum does not depend on R_m for $R_m \to \infty$. According to a general classification proposed by Molchanov, Ruzmaikin & Sokoloff (1985, figure 5*b*), dynamos of this type should be properly called *intermediate* rather than fast. As shown below (see also Gilbert 1988), the screw dynamo with a smooth velocity field is slow.

The relatively low value of the magnetic Reynolds number required for the field self-excitation, $R_{\mathrm{m\,cr}}$, is an attractive property of the screw dynamo. For piecewise constant distributions of the angular velocity and longitudinal velocity the minimal threshold value is as low as $R_{\mathrm{m\,cr}} \approx 15$–17 (Gailitis & Freiberg 1976, 1980). This high efficiency of generation makes promising laboratory realizations of the screw dynamo, either in a directed experiment (Gailitis, Freiberg & Lielausis 1977; see also Kirko 1985) or in bulk flows of liquid sodium in industrial devices, e.g. in breeder reactors (Kirko 1985).

On the other hand, an extremely simple form of the motion, which generates magnetic fields in this case, is widespread in nature and the screw dynamo may undoubtedly act in astrophysical objects. For example, large-scale screw motions are believed to be present in astrophysical jets – powerful plasma outflows from nuclei of galactic and extragalactic active objects (see, e.g. the review of Begelman, Blanford & Rees 1984). Magnetic fields provide the synchrotron radiation of the jets and, probably, their collimation and confinement. The screw dynamo may prove to be one of the main, if not the principal, source of magnetic fields in these spectacular active objects.

As shown by Lortz (1968) and Ponomarenko (1973), for the screw motion of a rigid cylinder ($\Omega = \mathrm{const}$, $V_z = \mathrm{const}$ for $r \leqslant r_0$ and $\Omega = 0$, $V_z = 0$ for $r > r_0$) the generated field is concentrated at the discontinuity of the velocities, $r = r_0$; the growth rate γ of any mode proves to be inversely proportional to the cubic root of the magnetic Reynolds number, $\gamma \propto R_{\mathrm{m}}^{-\frac{1}{3}}$. The dominant magnetic mode has, at the radius where the field is maximal, a helical magnetic line whose pitch is equal to the pitch of the streamlines, $m/k = -V_z/\Omega$, where m and k are the azimuthal and longitudinal wavenumbers of the field, respectively.

It is clear, however, that flows of viscous fluids are characterized by smooth velocity distributions $\Omega(r)$ and $V_z(r)$; consequently, the fields generated by a helical flow of a fluid can have properties radically differing from those for the case of rigid motion. The problem of the hydromagnetic screw dynamo is much more versatile from the physical point of view but it cannot be solved as simply as the rigid-body dynamo model. In this case an appropriate method of solution is provided by the asymptotic methods of fluid mechanics.

Here we solve the problem of the hydromagnetic screw dynamo with use of singular perturbation theory for large values of the magnetic Reynolds number; the asymptotic solutions that arise are either of the WKB or of the singular-layer type with power-law expansions in the asymptotic parameter (we call the latter *magnetic-layer asymptotics* bearing in mind their similarity to Prandtl's boundary layers in hydrodynamics). The induction equation contains a small coefficient R_{m}^{-1} at the Laplacian, which suggests that for $R_{\mathrm{m}} \gg 1$ the magnetic field concentrates within thin layers (or, possibly, ropes and even points for other flows – cf. Sokoloff, Shukurov & Ruzmaikin 1983). Hence, an optimal way to derive the solution is to employ singular perturbation theory. In hydromagnetic dynamo problems all three components of the magnetic field are essentially non-negligible, and we use extensions of the perturbation techniques appropriate to systems of equations (e.g. Maslov & Fedorjuk 1981). Experience with these methods indicates that one or two leading orders in the solution give quite realistic estimates of the eigenvalues (the field growth rates) even for values of the asymptotic parameter as low as 3–5 (cf. Baryshnikova & Shukurov 1987). In addition, it should be recognized that the perturbation series we deal with are of an asymptotic nature, i.e. they may be divergent. Thus, it seems wise to calculate only a few terms, given that the essential physics has been retained.

When magnetic configurations are highly concentrated in space, any homogeneous boundary conditions are easily fulfilled because of a sharp decrease of the field with distance from the singular layer. When the electrical conductivity is uniform, vanishing of the field at infinity (at least as r^{-2}) is the only relevant boundary condition. In the case of non-uniform conductivity, e.g. when the generation region is surrounded by an insulator, this boundary condition is an approximate one and is valid only until the size of the conductive region greatly exceeds the thickness of a layer where the magnetic field is concentrated. In principle, the technique of matched asymptotic expansions can be used to incorporate specific boundary conditions posed at a finite distance from the singular layer. As is shown by the numerical calculations of Solovyev (1985), the hydromagnetic screw dynamo is relatively insensitive to the choice of boundary conditions even for moderate Reynolds numbers.

We consider a cylindrically symmetric flow which is uniform along the symmetry axis. In some applications where the ratio of longitudinal to radial dimensions of the generation region is not very large, the effect of the finite length of the cylinder can be considerable. (In astrophysical jets this ratio is very large and the cylinder can be considered as infinite.) The ends of the cylinder can be described through the introduction of a weak dependence of the velocity field on the longitudinal coordinate; otherwise, one can consider a flow with streamlines winding around a torus of large radius (Lortz 1968). In the latter case the longitudinal boundary conditions reduce to the periodicity condition $kR = n$, where R is the torus' larger radius, k is the longitudinal wavenumber of the field, and n is an integer. This results in quantization of the longitudinal wavenumber.

2. Basic equations and asymptotic estimates

Consider the generation of a magnetic field by a fixed axisymmetric flow of incompressible conducting fluid with the velocity field given by

$$V = \{0, r\Omega(r), V_z(r)\} \tag{1}$$

in polar cylindrical coordinates $\{r, \phi, z\}$. Evolution of the magnetic field $\boldsymbol{H}$ is governed by the induction equation

$$\frac{\partial \boldsymbol{H}}{\partial t} = \nabla \times (V \times \boldsymbol{H}) + R_m^{-1} \nabla^2 \boldsymbol{H}, \tag{2}$$

where R_m is the magnetic Reynolds number. We use dimensionless quantities with the characteristic radius r_*, dependent on the chosen velocity field, as the unit length. When the angular and longitudinal velocities are measured in units of Ω_* and V_{z*}, respectively, the combination $r_*/(\Omega_*^2 r_*^2 + V_{z*}^2)^{\frac{1}{2}}$ becomes the unit of time. The magnetic Reynolds number is defined as

$$R_m = \frac{r_*(\Omega_*^2 r_*^2 + V_{z*}^2)^{\frac{1}{2}}}{\nu_m}.$$

Equation (2) is valid for a uniform magnetic diffusivity, ν_m.

Since the velocity field (1) is axisymmetric, homogeneous along the z-axis and time-independent, the normal modes of the generated magnetic field have the following form:

$$\boldsymbol{H} = \begin{bmatrix} H_r \\ H_\phi \\ H_z \end{bmatrix} \exp(\gamma t + \mathrm{i}m\phi + \mathrm{i}kz), \tag{3}$$

where the amplitudes H_r, H_ϕ, and H_z are functions of the radius, while the growth rate γ and the wavenumbers m and k are, of course, integer and real constants.

The components of the induction equation (2) are now given by

$$\gamma H_r + \mathrm{i}(m\Omega + kV_z) H_r = R_\mathrm{m}^{-1}\left\{\left[\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}}{\mathrm{d}r}\right) - \frac{m^2+1}{r^2} - k^2\right] H_r - \frac{2\mathrm{i}m}{r^2} H_\phi\right\}, \tag{4}$$

$$\gamma H_\phi + \mathrm{i}(m\Omega + kV_z) H_\phi = r\frac{\mathrm{d}\Omega}{\mathrm{d}r} H_r + R_\mathrm{m}^{-1}\left\{\left[\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}}{\mathrm{d}r}\right) - \frac{m^2+1}{r^2} - k^2\right] H_\phi + \frac{2\mathrm{i}m}{r^2} H_r\right\}, \tag{5}$$

$$\gamma H_z + \mathrm{i}(m\Omega + kV_z) H_z = \frac{\mathrm{d}V_z}{\mathrm{d}r} H_r + R_\mathrm{m}^{-1}\left\{\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}}{\mathrm{d}r}\right) - \frac{m^2}{r^2} - k^2\right\} H_z. \tag{6}$$

We note immediately that the z-component of the field does not enter the first two equations, (4) and (5), and they can be solved autonomously. We stress, however, that this does not imply that a two-dimensional field with $H_z \equiv 0$ can be generated. For non-vanishing H_r equation (6) has only non-trivial solutions for H_z. Below, we show that the components H_r and H_ϕ have differing orders of magnitude in the asymptotic parameter R_m and, therefore, the solenoidality of the field $\nabla \boldsymbol{H} = 0$ requires the presence of H_z. Note also that equation (6) is fulfilled exactly for H_z derived from $\nabla \boldsymbol{H} = 0$ and for H_r and H_ϕ obtained from (4) and (5) (Solovyev 1985). Thus, after H_r and H_ϕ have been derived from (4) and (5), H_z can be easily found from the solenoidality condition as

$$H_z = \frac{\mathrm{i}}{kr}\frac{\mathrm{d}}{\mathrm{d}r}(rH_r) - \frac{m}{kr} H_\phi.$$

Let us begin with a qualitative analysis of the system (4)–(6). We distinguish the terms essential for the field generation. First, in (4) the last term in the braces is the source for the radial field component. When $m = 0$, it vanishes and the field decays in accordance with Cowling's theorem. This term also vanishes for finite m and $\nu_\mathrm{m} = 0$, hence we expect that $\mathrm{Re}\,\gamma \to 0$ for $R_\mathrm{m} \to \infty$, i.e. the considered dynamo is a slow one.

The term including $r\,\mathrm{d}\Omega/\mathrm{d}r$ on the right-hand side of (5), which describes the drawing out of the radial field in the azimuthal direction by the differential rotation, is also essential for the field amplification. The field would decay without it. It is known, however, that the differential rotation and magnetic diffusion alone are not sufficient for an infinitely long maintenance of the field (see Moffatt 1978). Therefore, we can see in advance that the advection terms featuring V_z on the left-hand sides of (4)–(6) are of vital importance for the screw dynamo.

The radial part of the Laplace operator, $(1/r)(\mathrm{d}/\mathrm{d}r)(r\,\mathrm{d}/\mathrm{d}r)$, ensures decay of the generated field at $r \to \infty$; its presence is necessary for fulfilment of the boundary conditions. Of course, the terms that include γ are also necessary.

Construction of the asymptotic solution is based on the requirement that all the above-mentioned terms be present in the lowest-order equations of the perturbation theory.

Let the characteristic radial scale of the field be such that the symbolic asymptotic estimate $\mathrm{d}/\mathrm{d}r \sim R_{\mathrm{m}}^{\delta}$ holds, with δ still unknown. Then

$$R_{\mathrm{m}}^{-1}\,\mathrm{d}^2H_r/\mathrm{d}r^2 \sim R_{\mathrm{m}}^{2\delta-1}H_r.$$

The field generation would be effective only when the latter term and the source $R_{\mathrm{m}}^{-1}\,2\mathrm{i}mH_\phi/r^2$ on the right-hand side of (4) are of the same order in R_{m}, i.e. when $R_{\mathrm{m}}^{2\delta}H_r \sim H_\phi$. Turning now to (5), we obtain $H_r \sim R_{\mathrm{m}}^{2\delta-1}H_\phi$ from the asymptotic equality $H_r\, r\,\mathrm{d}\Omega/\mathrm{d}r \sim R_{\mathrm{m}}^{-1}\,\mathrm{d}^2H_\phi/\mathrm{d}r^2$. Hence, we obtain $2\delta-1 = -2\delta$, i.e. $\delta = \frac{1}{4}$.

Therefore,

$$\frac{H_r}{H_\phi} \sim R_{\mathrm{m}}^{-\frac{1}{2}}, \quad \frac{\mathrm{d}}{\mathrm{d}r} \sim R_{\mathrm{m}}^{\frac{1}{4}}.$$

Equation (6) now gives $$H_z \sim R_{\mathrm{m}}^{\frac{1}{2}}H_r \sim H_\phi.$$

Next, the terms on the left-hand sides of (4)–(6) are of the same order in R_{m} as the essential right-hand-side terms only when $\gamma + \mathrm{i}(m\Omega + kV_z) \sim R_{\mathrm{m}}^{-\frac{1}{2}}$, which means that

$$\operatorname{Re}\gamma \sim R_{\mathrm{m}}^{-\frac{1}{2}}, \tag{7a}$$

$$\operatorname{Im}\gamma + m\Omega + kV_z \sim R_{\mathrm{m}}^{-\frac{1}{2}}. \tag{7b}$$

Equation 7(*a*) gives the asymptotic order of the growth rate. It should be remembered that we consider solutions whose radial scale is much smaller than the characteristic flow scales. For this reason our solutions do not describe the case of the rigid cylinder where the velocity profiles are discontinuous (formally, of zero characteristic scale) and for which $\gamma \sim R_{\mathrm{m}}^{-\frac{1}{3}}$. It is clear from the discussion above that the approximation of a rigid cylinder can be applied to realistic velocity profiles only when the velocity jump is smeared over a radius interval not exceeding $r_* R_{\mathrm{m}}^{-\frac{1}{4}}$ in width; otherwise, the asymptotic forms obtained here are applicable. Presuming that the thickness of the layer where the velocity can suffer a jump is of the order of $r_* Re^{-\frac{1}{2}}$ (where Re is the hydrodynamic Reynolds number), we obtain as a necessary condition for applicability of the rigid-body screw dynamo model that $Re^2 < R_{\mathrm{m}}$. Note also that one can envisage velocity distributions with a sharp boundary, for which the hydromagnetic asymptotics are applicable near the axis simultaneously with an additional generation region at the velocity jump at the boundary.

Asymptotic estimate (7*b*) has a nature entirely different from all preceding estimates. Indeed, Ω and V_z do not depend on R_{m} at all and their combination $\operatorname{Im}\gamma + m\Omega + kV_z$ can be small with the required accuracy only at some (if any) points (note that $\operatorname{Im}\gamma$, m and k are constants while Ω and V_z are functions). Hence, (7*b*) is actually an equation that determines the ratio of the wavenumbers, m and k, of the generated field when the position of the singular layer, $r = r_0$, has been specified.

Two cases can be distinguished in which asymptotic solutions have distinct forms. In the simplest case when Ω and V_z show exactly the same dependence on r, i.e. when the pitch of a helical streamline, $V_z(r)/\Omega(r)$, does not depend on radius, certain m and k always exist such that

$$\frac{m}{k} = -\frac{V_z(r)}{\Omega(r)} = \text{const.} \tag{8}$$

and the advection term vanishes identically at all radii simultaneously while $\operatorname{Im}\gamma = O(R_{\mathrm{m}}^{-\frac{1}{2}})$. Expression (8) determines the ratio of the azimuthal and longitudinal wavenumbers of the field generated by such flow; the position of the field maximum will be found in the next section.

Note that when (8) is fulfilled the longitudinal velocity V_z does not enter explicitly the dynamo equations (4) and (5). However, this does not mean that the longitudinal velocity is inessential for the generation. When $V_z \to 0$, equation (8) gives $m = 0$. Meanwhile, we know from Cowling's theorem that the axisymmetric field can only decay in the velocity field considered. Otherwise, for $k \to \infty$ the solenoidality condition and the induction equation, namely its z-component, are incompatible, which also indicates the impossibility of field growth.

For another case, viz. when Ω and V_z have different radial distributions, expand $\Omega + (\mathrm{Im}\,\gamma + kV_z)/m$ in a Taylor series near the radius $r = r_0$ where the field concentrates:

$$\frac{1}{m}\,\mathrm{Im}\,\gamma + \Omega + \frac{k}{m}\,V_z = q_0 + q_1(r-r_0) + q_2(r-r_0)^2 + \ldots,$$

where q_j are the corresponding derivatives with respect to r evaluated at $r = r_0$ that are of order unity and do not depend on R_m. Meanwhile, matching of the orders of magnitude of the generation terms in (4)–(6) requires that (7) is fulfilled in the vicinity $r - r_0 = O(R_\mathrm{m}^{-\frac{1}{4}})$ of the radius r_0 where the field concentrates. Hence, the generated field has a configuration such that $q_0 = 0$ and $q_1 = 0$. The remaining terms in the latter expansion are of the required order $(r-r_0)^2 \sim R_\mathrm{m}^{-\frac{1}{2}}$. Thus, the procedure of construction of the asymptotic solution now is as follows. For any given radius r_0 we obtain the ratio of wavenumbers from

$$m\Omega'(r_0) + kV_z'(r_0) = 0,$$

where the prime denotes derivatives in r. For a given m, we derive $\mathrm{Im}\,\gamma$ from

$$\mathrm{Im}\,\gamma = -m\Omega(r_0) - kV_z(r_0).$$

Note that there exists one particular magnetic mode for which the pitch of magnetic lines equals the pitch of the streamlines at this radius:

$$\frac{m}{k} = -\frac{V_z(r_0)}{\Omega(r_0)},$$

and $\mathrm{Im}\,\gamma = O(R_\mathrm{m}^{-\frac{1}{2}})$ for that mode which concentrates at the radius determined by

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{V_z}{\Omega}\right)\bigg|_{r=r_0} = 0,$$

i.e. at the position where the pitch of the streamlines has an extremum.

In the vicinity of the singular layer we have an expansion

$$\frac{1}{m}\,\mathrm{Im}\,\gamma + \Omega + \frac{k}{m}\,V_z = q_2(r-r_0)^2 + O[(r-r_0)^3] \sim R_\mathrm{m}^{-\frac{1}{2}}$$

since $r - r_0 \sim R_\mathrm{m}^{-\frac{1}{4}}$ in the region of effective generation. This asymptotic estimate should not be interpreted as implying a dependence of the hydrodynamic flow pattern on electroconductive properties of the fluid. This estimate merely means that magnetic field generation proceeds most effectively within a thin cylindrical layer $r - r_0 \sim R_\mathrm{m}^{-\frac{1}{4}}$ where the term $\mathrm{Im}\,\gamma + m\Omega + kV_z$ has the required small value $O(R_\mathrm{m}^{-\frac{1}{2}})$.†

† An expansion of the velocity field in powers of the magnetic Reynolds number is also included in the well-known dynamo of Braginsky (1964). Braginsky's expansion has a global character, $V = V_0(r) + \epsilon V_1(r) + \ldots$ (localization in the velocity space!), and it follows from the quasi-stationary dynamo equations that self-excitation of the field requires $\epsilon \sim R_\mathrm{m}^{-\frac{1}{2}}$ (Moffatt 1978). In this case the field is not necessarily strongly localized in space.

Thus, the parameter for our asymptotic expansions is $R_m^{-\frac{1}{4}}$ and they are applicable when $R_m^{\frac{1}{4}} \gg 1$ rather than when $R_m \gg 1$, the latter being a less stringent restriction. Correspondingly, the accuracy of the following asymptotic estimates is related to the value of $R_m^{-\frac{1}{4}}$.

In this section we have shown that the nature of asymptotic solutions for magnetic fields in a laminar helical flow crucially depends on whether Ω and V_z are identically or differently distributed along the radius. Consider these cases separately.

3. Identical profiles of $\Omega(r)$ and $V_z(r)$

The field generated in the case of identical profiles of the angular and longitudinal velocities has azimuthal and longitudinal wavenumbers such that equality (8) holds and the advection term identically vanishes in the induction equation. In accordance with estimates of §2, we seek an asymptotic solution of the WKB type:

$$\left.\begin{aligned} \gamma &= R_m^{-\frac{1}{2}}(\gamma_0 + R_m^{-\frac{1}{4}}\gamma_1 + R_m^{-\frac{1}{2}}\gamma_2 + \ldots), \\ H_r &= R_m^{-\frac{1}{2}}(h_{0r} + R_m^{-\frac{1}{4}}h_{1r} + \ldots) \exp[\mathrm{i}R_m^{\frac{1}{4}} S(r)], \\ H_\phi &= (h_{0\phi} + R_m^{-\frac{1}{4}}h_{1\phi} + \ldots) \exp[\mathrm{i}R_m^{\frac{1}{4}} S(r)], \end{aligned}\right\} \tag{9}$$

where h_{jr} and $h_{j\phi}$ are functions of the radius r. For large magnetic Reynolds numbers, the growth rate and configuration of the generated field are determined primarily by the lowest-order solution, i.e. by γ_0, S, h_{0r}, $h_{0\phi}$ and h_{0z}. However, the critical value of the magnetic Reynolds number that corresponds to $\mathrm{Re}\,\gamma = 0$ can be estimated only when higher-order approximations for the eigenvalue are evaluated.

Before proceeding to the evaluation of the asymptotic solution, we discuss some general properties of the WKB method as compared with other asymptotic methods. An asymptotic solution of a given problem often can be reached by different asymptotic methods; forms of the solutions may differ markedly even though they are, of course, asymptotically equivalent to each other. For instance, solutions described in this section can be obtained both by the WKB method and by the boundary-layer expansion. Following an appeal 'Let all the flowers bloom', we choose the WKB approach in this section; §4 describes the application of boundary-layer expansions to our problem.

The WKB method has an important advantage: in its framework the derivation of a solution reduces to a sequence of standard steps that always results in an explicit form of the solution to any order (an example is the WKB solution of three-dimensional mean-field dynamo problems by Sokoloff *et al.* 1983 and Ruzmaikin, Sokoloff & Starchenko 1988*b*). In other asymptotic methods approximate equations of any order can also be derived without any difficulty but their solution often represents a problem no less complicated than the original one.

The WKB method has a well-known difficulty: its application becomes very complicated near turning points where the amplitude factor ($\boldsymbol{h}_0 + R_m^{-\frac{1}{4}}\boldsymbol{h}_1 + \ldots$ in our case) is singular. This difficulty restricts application of this method in quantum mechanics and plasma physics. Recently V. P. Maslov has developed a general method of derivation of the WKB solution at turning points of arbitrary kind (see e.g. Maslov & Fedorjuk 1981). It has become clear simultaneously that problems of quantum mechanics and wave propagation, where highly excited states are of principal importance, involve many more complications in the application of the WKB method than, for example, dynamo theory, where the basic and low states are in focus. The frequently met opinion that the WKB method is always inapplicable to the analysis of weakly excited states is based on a misunderstanding associated with an unfortunate particular scheme of calculations adopted in some textbooks. A mathematically rigorous procedure of treatment of low states is called 'the oscillatory approximation' in quantum mechanics.

From a formal point of view, the wide applicability of the WKB method in dynamo problems is due to the fact that, in situations that are interesting for dynamo theory, both turning points merge. More precisely, they reside within the region of concentration of eigenfunctions of moderately excited states. For the asymptotic forms obtained here this is true for eigenmodes with $n \gtrsim R_m^{\frac{1}{4}}$ (see below for the definition of the level number n). Therefore, the rule of passing through turning points reduces to a simple and readily accessible requirement of regularity of an eigenfunction at its maximum. These arguments are applicable when the effective potential is a quadratic function of coordinates near the bottom of a potential well.

We should note that difficulties similar to those of the WKB method must arise in any asymptotic method (in the boundary-layer expansions, in particular) when highly excited levels are considered. Indeed, all parameters of the problem (including the level number) must not exceed a (large) asymptotic parameter.

Now we turn to an evaluation of the WKB asymptotic forms (9). Substitution of the asymptotic expansions into (4) and (5) and combination of the terms with common powers of R_m gives the following equations in the lowest two orders:

$$\left.\begin{aligned}(\gamma_0+S'^2)\,h_{0r}+\frac{2im}{r^2}h_{0\phi}&=0,\\ -r\Omega' h_{0r}+(\gamma_0+S'^2)\,h_{0\phi}&=0,\end{aligned}\right\}\tag{10}$$

$$\left.\begin{aligned}(\gamma_0+S'^2)\,h_{1r}+\frac{2im}{r^2}h_{1\phi}&=\left(-\gamma_1+\frac{iS'}{r}+iS''+2iS'\frac{d}{dr}\right)h_{0r},\\ -r\Omega' h_{1r}+(\gamma_0+S'^2)\,h_{1\phi}&=\left(-\gamma_1+\frac{iS'}{r}+iS''+2iS'\frac{d}{dr}\right)h_{0\phi}.\end{aligned}\right\}\tag{11}$$

The homogeneous system of algebraic equations (10) for h_{0r} and $h_{0\phi}$ has non-trivial solutions only when its determinant vanishes, which brings us to the Hamilton–Jacobi equation:

$$\gamma_0=\pm(1-i)\left(\frac{m\Omega'}{r}\right)^{\frac{1}{2}}-S'^2,\tag{12}$$

for the action S and the lowest approximation to the eigenvalue, γ_0. The prime denotes derivatives in r.

We expect that the solution concentrates near a certain radius $r=r_0$. Therefore, the action $S(r)$ must have the maximum at this point, i.e. $S'(r_0)=0$. In addition, decrease of the magnetic field strength with distance from the singular layer is guaranteed by $\operatorname{Im} S(r)>0$ for $r>0$ (of course, $\operatorname{Im} S(0)=0$).

Since $S'=0$ at $r=r_0$, equation (12) immediately relates γ_0 to the value of $m\Omega'/r$ at this point. It is clear that the singular layer must be found at that position where $\operatorname{Re}\gamma_0$ attains its maximal possible value, which implies that the right-hand side of (12) is maximal at the point where the field concentrates:

$$\left.\frac{d}{dr}\left(\frac{1}{r}\frac{d\Omega}{dr}\right)\right|_{r=r_0}=0.\tag{13}$$

These arguments can be exposed at another, more formal level. The right-hand side of (12) is the Hamiltonian,

$$\mathscr{H}(r,p) = \pm(1-\mathrm{i})[m\Omega'(r)/r]^{\frac{1}{2}} - p^2,$$

where $p \equiv S'$. The point (r_0, p_0) where the solution concentrates is the singular point of the Hamiltonian, i.e.

$$\frac{\partial \mathscr{H}}{\partial r} = 0, \quad \frac{\partial \mathscr{H}}{\partial p} = 0 \quad \text{at} \quad (r,p) = (r_0, p_0).$$

This gives $p_0 = 0$ and equation (13) for the position of the singular layer while stability of the singular point requires that r_0 is the maximum point of Ω'/r when the plus sign is chosen in (12). Now taking the Hamiltonian–Jacobi equation (12) at the singular point of the Hamiltonian we obtain the eigenvalue† as

$$\gamma_0 = (1 \mp \mathrm{i})\left(\frac{m|\Omega_0'|}{r_0}\right)^{\frac{1}{2}}, \tag{14}$$

where we have chosen the branches with $\mathrm{Re}\,\gamma_0 > 0$, the minus (plus) sign corresponds to $\Omega_0' > 0$ ($\Omega_0' < 0$), and the subscript 0 refers to the singular point $r = r_0$.

Note that r_0 determined by (13) differs from zero only for non-monotonic dependencies of Ω' on r. For monotonic dependencies and for $|\Omega'|$ growing slower than r we obtain $r_0 = 0$. However, all realistic velocity distributions have $\Omega'(0) = 0$ and $V_z'(0) = 0$ which means that (13) and (14) have no singularity at $r_0 = 0$. We should also note that one can envisage velocity distributions for which (13) has no solutions. In this case the field generation, if possible at all, must proceed in a different manner from that prescribed by WKB asymptotic solutions, e.g. with formation of the magnetic layers (see §4).

Now substitute (14) into (12) to obtain the following solution for the action:

$$S = \mp(2m)^{\frac{1}{4}} \exp\left(\pm\frac{\mathrm{i}\pi}{8}\right) \int_{r_0}^{r} \left[\left(\frac{|\Omega'|}{r}\right)^{\frac{1}{2}} - \left(\frac{|\Omega_0'|}{r_0}\right)^{\frac{1}{2}}\right]^{\frac{1}{2}} \mathrm{d}r, \tag{15}$$

where the minus sign corresponds to $\Omega_0' > 0$ and the plus to $\Omega_0' < 0$. In both cases those branches of the function S are chosen that correspond to $\mathrm{Re}\,\gamma_0 > 0$ and $\mathrm{Im}\,S > 0$.

For practical purposes, it is sufficient to expand the function $|\Omega'(r)|/r$ in a Taylor series in the vicinity of the singular point,

$$\frac{|\Omega'|}{r} = \frac{|\Omega_0'|}{r_0} + g_1(r-r_0)^2 + \dots,$$

which gives

$$S = \pm\tfrac{1}{2}\exp\left(\pm\frac{\mathrm{i}\pi}{8}\right)\left(\frac{m g_1^2 r_0}{2|\Omega_0'|}\right)^{\frac{1}{4}} [(r-r_0)^2 + O(r-r_0)^3]. \tag{16}$$

Thus we have obtained both the zero-order approximation to the field growth rate, γ_0, and the action, S, from the Hamilton–Jacobi equation (12). Now, the zero-order equations (10) give the ratio of the field components,

$$\frac{h_{0r}}{h_{0\phi}} = (\pm 1 - \mathrm{i})\left(\frac{m}{r^3|\Omega'|}\right)^{\frac{1}{2}},$$

† In terms of quantum mechanics, a rough estimate (14) of the lowest energy level is simply the depth of the potential well for the potential $\pm(1-\mathrm{i})(m\Omega'/r)^{\frac{1}{2}}$.

rather than the separate components since the determinant of the system (10) vanishes. In order to determine, for example, $h_{0\phi}$ one should use the first-order equations (11) which simultaneously yield γ_1 in the following way.

The matrix of the left-hand sides of the system (11) for h_{1r} and $h_{1\phi}$ coincides with the degenerate matrix of the zero-order equations (10). It is well known that the degenerate inhomogeneous system of linear algebraic equations (11) has non-trivial solutions only when the vector of the right-hand sides, $\hat{\boldsymbol{L}}_1 \boldsymbol{h}_0$, is orthogonal to the eigenvector $\boldsymbol{h}_0^*$ of the adjoint matrix of the left-hand sides. The matrix $\hat{\boldsymbol{L}}_0^*$ adjoint to the matrix $\hat{\boldsymbol{L}}_0$ of (10) has the form

$$\hat{\boldsymbol{L}}_0^* = \begin{bmatrix} \gamma_0 + S'^2 & -r\Omega' \\ -\dfrac{2\mathrm{i}m}{r^2} & \gamma_0 + S'^2 \end{bmatrix},$$

and the ratio of the components of its eigenvector is given by

$$\frac{h_{0r}^*}{h_{0\phi}^*} = \frac{\pm 1 - \mathrm{i}}{2} \left(\frac{r^3 |\Omega'|}{m} \right)^{\frac{1}{2}}.$$

Orthogonality of the vectors $\boldsymbol{h}_0^*$ and $\hat{\boldsymbol{L}}_1 \boldsymbol{h}_0$, i.e. $(\boldsymbol{h}_0^* . \hat{\boldsymbol{L}}_1 \boldsymbol{h}_0) = 0$, now brings us to the following equation for γ_1 and $h_{0\phi}$:

$$2\mathrm{i}S' h'_{0\phi} + \left[-\gamma_1 + \mathrm{i}S'' + \frac{\mathrm{i}S'}{r} - \frac{\mathrm{i}S'}{2} \frac{\mathrm{d}}{\mathrm{d}r} \ln (r^3 \Omega') \right] h_{0\phi} = 0. \tag{17}$$

It is sufficient to solve this equation in the vicinity of the singular point. As follows from (16), for $r \to r_0$ we have $S' \sim (r - r_0)$ and $S'' \sim$ const. Therefore, when r tends to r_0 for $r_0 \neq 0$ the last two terms in the square brackets vanish, $S'/r \to 0$ and $S' d[\ln (r^3 \Omega')]/\mathrm{d}r \to 0$. However, for $r_0 = 0$ these terms acquire non-vanishing values $S'/r \to S''(0)$ and $\mathrm{d}[\ln (r^3 \Omega')]/\mathrm{d}r \to 4/r$. Hence, for $|r - r_0| \ll 1$ equation (17) takes the form

$$2\mathrm{i}S''(r_0)(r - r_0) h'_{0\phi} + [-\gamma_1 + \mathrm{i}S''(r_0)] h_{0\phi} = 0 \quad \text{for } r_0 \neq 0$$

and

$$2\mathrm{i}S''(0)\, r h'_{0\phi} - \gamma_1 h_{0\phi} = 0 \quad \text{for } r_0 = 0.$$

Solutions to these equations are homogeneous functions of $(r - r_0)$,

$$h_{0\phi} = (r - r_0)^n, \tag{18}$$

where $n = 0, 1, 2, \ldots$ for $r_0 \neq 0$. However, for $r_0 = 0$ we should take into account that the azimuthal field must vanish at the symmetry axis, i.e. $n > 0$. Moreover, we have $h_{0r} \propto (r^3 \Omega')^{-\frac{1}{2}} h_{0\phi}$ while $\Omega' \propto r$ for $r \to 0$. Consequently, for $r_0 = 0$, h_{0r} is bound at the axis only when $n \geqslant 2$. Therefore, for $r_0 = 0$ in (18) we have $n = 2, 3, \ldots$.

Using (18) we obtain now

$$\gamma_1 = \mathrm{i}(2n + 1) S''(r_0) \quad \text{for } r_0 \neq 0 \quad \text{with } n = 0, 1, 2, \ldots \tag{19}$$

and

$$\gamma_1 = 2\mathrm{i}n S''(0) \quad \text{for } r_0 = 0 \quad \text{with } n = 2, 3, \ldots. \tag{20}$$

Knowing γ_0 and γ_1 we can give a crude estimate of the critical value of the magnetic Reynolds number, which corresponds to $\mathrm{Re}\,\gamma = 0$:

$$R_{\mathrm{m\,cr}}^{\frac{1}{4}} \sim -\frac{\mathrm{Re}\,\gamma_1}{\mathrm{Re}\,\gamma_0} \quad \text{when } \mathrm{Re}\,\gamma_1 < 0$$

(for $\mathrm{Re}\,\gamma_1 > 0$ the next approximation, γ_2, must be used). Since $|\mathrm{Re}\,\gamma_1|$ increases with n, while $\mathrm{Re}\,\gamma_0$ decreases, the mode with the lowest possible value of n, either 0 or 2, has the lowest generation threshold and the largest growth rate. Since the terms neglected in the estimation of $R_{\mathrm{m\,cr}}$ are of order $R_{\mathrm{m}}^{-\frac{1}{4}}$, one can expect that the relative accuracy of this estimate is of this order.

The reliability of estimates of the critical magnetic Reynolds number from asymptotic expansions of the eigenvalue crucially depends on the resulting value: the larger $R_{\mathrm{m\,cr}}$ is, the more accurate is the estimate. Confidence in such estimates can be acquired through comparison with numerical examples. For instance, such a comparison was made by Ruzmaikin *et al.* (1988*a*) for a screw dynamo in Couette–Poiseuille flow. Asymptotic results obtained here agree reasonably well with numerical calculations and justify this way of obtaining crude estimates of the critical magnetic Reynolds number. Such a comparison of similar asymptotic and numerical results for related dynamo problems is also discussed by Baryshnikova & Shukurov (1987).

Note, however, that the last expression for $R_{\mathrm{m\,cr}}$ decreases with m since $\mathrm{Re}\,\gamma_0 \propto m^{\frac{1}{2}}$ and $\mathrm{Re}\,\gamma_1 \propto m^{\frac{1}{4}}$. The reason for this unphysical behaviour of $R_{\mathrm{m\,cr}}$ is that the two lowest perturbation orders considered do not include the terms proportional to m^2 in the induction equation, which describe the azimuthal diffusion of the magnetic field. These terms appear only in the second approximation for the eigenvalue, γ_2. The procedure of derivation of the next approximation is principally the same as that carried out above for the first approximation. Here we give the result (see Sokoloff, Shukurov & Shumkina 1989):

$$\gamma_2 = \tfrac{1}{2}(r^3\Omega')^{\frac{1}{2}}\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left[r\frac{\mathrm{d}}{\mathrm{d}r}(r^3\Omega')^{-\frac{1}{2}}\right]\Bigg|_{r=r_0} - \frac{m^2+1}{r_0^2} - k^2 \quad \text{for } r_0 \neq 0, \tag{21}$$

while $$\gamma_2 = -k^2 = -m^2\Omega_0^2/V_{z0}^2 \quad \text{for } r_0 = 0,$$

independently of n. The following estimate for the critical magnetic Reynolds number comprises all relevant physical effects:

$$R_{\mathrm{m\,cr}}^{\frac{1}{4}} \sim -\frac{1}{2}\frac{\mathrm{Re}\,\gamma_1}{\mathrm{Re}\,\gamma_0} - \left[\frac{1}{4}\left(\frac{\mathrm{Re}\,\gamma_1}{\mathrm{Re}\,\gamma_0}\right)^2 - \frac{\mathrm{Re}\,\gamma_2}{\mathrm{Re}\,\gamma_0}\right]^{\frac{1}{2}}. \tag{22}$$

The relative accuracy of this estimate is again expected to be $O(R_{\mathrm{m}}^{-\frac{1}{4}})$.

Thus we have concluded a derivation of the asymptotic solution for the case of identical profiles, $\Omega(r)/V_z(r) = \text{const}$, having obtained the configuration and growth rate of the magnetic field and estimated the critical value of the magnetic Reynolds number. Examples of specific velocity distributions are considered in §5.

Above, we have obtained solutions for which $\mathrm{Im}\,\gamma = O(R_{\mathrm{m}}^{-\frac{1}{2}})$ and m and k are related by (8). Situations are conceivable when k is prescribed by some external factors, e.g. by boundary conditions in the z-coordinate. In such cases (8) cannot be fulfilled since m is an integer. In other words, the advection term in the induction equation cannot vanish at all radii simultaneously, notwithstanding the fact that Ω and V_z have similar distributions. Therefore, the asymptotic forms that arise in these cases differ from the WKB type and are similar to the magnetic-layer asymptotics discussed in the next section. It is clear that in this case the field growth rate is smaller than that for freely growing modes with m and k obeying (8).

4. Distinct profiles of $\Omega(r)$ and $V_z(r)$

As shown in §2, in the case of non-identical profiles of the angular and longitudinal velocities the advection term in the induction equation never vanishes identically but the term $q_2 R_m^{\frac{1}{2}}(r-r_0)^2$ is inherited. The presence of this term in the induction equation precludes the WKB-type asymptotic forms. Thus, asymptotic forms that arise now are of another nature, of the magnetic-layer type.

Introduce the new stretched variable

$$x = (r-r_0)R_m^{\frac{1}{4}}$$

and consider the Prandtl type of singular layer (cf. Van Dyke 1975):

$$\gamma = R_m^{-\frac{1}{2}}(\gamma_0 + R_m^{-\frac{1}{4}}\gamma_1 + \ldots),$$

$$H_r = R_m^{-\frac{1}{2}}[h_{0r}(x) + R_m^{-\frac{1}{4}}h_{1r}(x) + \ldots],$$

$$H_\phi = h_{0\phi}(x) + R_m^{-\frac{1}{4}}h_{1\phi}(x) + \ldots.$$

Now expand the functions $\Omega + kV_z/m + \operatorname{Im}\gamma/m$ and $r\,\mathrm{d}\Omega/\mathrm{d}r$ in a Taylor series about the radius $r = r_0$:

$$\Omega + \frac{kV_z + \operatorname{Im}\gamma}{m} = q_2 x^2 R_m^{-\frac{1}{2}} + q_3 x^3 R_m^{-\frac{3}{4}} + \ldots,$$

$$r\frac{\mathrm{d}\Omega}{\mathrm{d}r} = G_0 + G_1 x R_m^{-\frac{1}{4}} + \ldots.$$

Here we consider the case when $r_0 \neq 0$, i.e. when within the singular layer the value of r^{-2} is not as large as $R_m^{\frac{1}{2}}$. The case $r_0 = 0$ requires much more cumbersome calculations but does not provide any new physical ideas.

The asymptotic forms given above are inserted into (4) and (5) and the coefficients at the various powers of $R_m^{-\frac{1}{4}}$ are equated to zero. The result is the following equations:

$$\left.\begin{aligned} &\gamma_0 h_{0r} + imq_2 x^2 h_{0r} - \frac{\mathrm{d}^2 h_{0r}}{\mathrm{d}x^2} + \frac{2im}{r_0^2}h_{0\phi} = 0,\\ &\gamma_0 h_{0\phi} + imq_2 x^2 h_{0\phi} - \frac{\mathrm{d}^2 h_{0\phi}}{\mathrm{d}x^2} - G_0 h_{0r} = 0 \end{aligned}\right\} \qquad (23)$$

to the zeroth order; and

$$\left.\begin{aligned} &\gamma_0 h_{1r} + imq_2 x^2 h_{1r} - \frac{\mathrm{d}^2 h_{1r}}{\mathrm{d}x^2} + \frac{2im}{r_0^2}h_{1\phi} = -(\gamma_1 + imq_3 x^3)\,h_{0r} + \frac{1}{r_0}\frac{\mathrm{d}h_{0r}}{\mathrm{d}x} + \frac{4im}{r_0^3}xh_{0\phi},\\ &\gamma_0 h_{1\phi} + imq_2 x^2 h_{1\phi} - \frac{\mathrm{d}^2 h_{1\phi}}{\mathrm{d}x^2} - G_0 h_{1r} = -(\gamma_1 + imq_3 x^3)\,h_{0\phi} + \frac{1}{r_0}\frac{\mathrm{d}h_{0\phi}}{\mathrm{d}x} + G_1 x h_{0r} \end{aligned}\right\} \qquad (24)$$

to the first order.

An exact solution to the zero-order equations (23) can be found as follows. Introduce new variables

$$\tilde{x} = (imq_2)^{\frac{1}{4}}x, \quad \tilde{\gamma} = (imq_2)^{-\frac{1}{2}}\gamma_0,$$

$$\tilde{h}_r = \frac{r_0}{(2m)^{\frac{1}{2}}}h_{0r}, \quad \tilde{h}_\phi = \left(\frac{i}{G_0}\right)^{\frac{1}{2}}h_{0\phi}.$$

In terms of these variables, (23) recasts as

$$\tilde{\gamma}\tilde{h}_r + \tilde{x}^2\tilde{h}_r - \frac{\mathrm{d}^2\tilde{h}_r}{\mathrm{d}\tilde{x}^2} + R\tilde{h}_\phi = 0,$$

$$\tilde{\gamma}\tilde{h}_\phi + \tilde{x}^2\tilde{h}_\phi - \frac{\mathrm{d}^2\tilde{h}_\phi}{\mathrm{d}\tilde{x}^2} - R\tilde{h}_r = 0,$$

where

$$R = \left(\frac{2G_0}{q_2 r_0^2}\right)^{\frac{1}{2}}.$$

This system reduces to a pair of uncoupled stationary Schrödinger equations for $h_+ = \tilde{h}_r + \mathrm{i}\tilde{h}_\phi$ and $h_- = \tilde{h}_r - \mathrm{i}\tilde{h}_\phi$:

$$\frac{\mathrm{d}^2 h_\pm}{\mathrm{d}\tilde{x}^2} - (\tilde{\gamma} \mp \mathrm{i}R)\, h_\pm - \tilde{x}^2 h_\pm = 0,$$

which has the well-known solutions

$$\left.\begin{aligned} \tilde{h}_r &= H_n(\tilde{x}) \exp\left(-\frac{\tilde{x}^2}{2}\right), \\ \tilde{h}_\phi &= \mp \mathrm{i}\tilde{h}_r, \\ \tilde{\gamma} &= -(2n+1) \pm \mathrm{i}R, \quad n = 0, 1, 2, \ldots, \end{aligned}\right\} \tag{25}$$

where $H_n(r)$ are Hermite polynomials.

Now we separate those branches of solutions that, first, decay at infinity and, second, have $\mathrm{Re}\,\gamma_0 > 0$. For different combinations of the signs of G_0 and q_2 the result is the following:

(i) When G_0 and q_2 have identical signs,

$$\gamma_0 = (\pm 1 + \mathrm{i}) \left|\frac{mq_2}{2}\right|^{\frac{1}{2}} [-(2n+1) + \mathrm{i}R], \tag{26}$$

and

$$h_{0r} = \frac{(2m)^{\frac{1}{2}}}{r_0} \exp\left[-\frac{1 \pm \mathrm{i}}{2}\left|\frac{mq_2}{2}\right|^{\frac{1}{2}} x^2\right] H_n[(\mathrm{i}mq_2)^{\frac{1}{4}}x],$$

$$h_{0\phi} = -(\pm 1 + \mathrm{i}) \left|\frac{G_0}{2}\right|^{\frac{1}{2}} \exp\left[-\frac{1 \pm \mathrm{i}}{2}\left|\frac{mq_2}{2}\right|^{\frac{1}{2}} x^2\right] H_n[(\mathrm{i}mq_2)^{\frac{1}{4}}x],$$

where the upper (lower) sign corresponds to positive (negative) G_0 and q_2; in addition, the signs of $h_{0\phi}$ and R can be taken negative in accordance with (25).

(ii) When G_0 and q_2 have different signs,

$$\gamma_0 = (\pm 1 + \mathrm{i}) \left|\frac{mq_2}{2}\right|^{\frac{1}{2}} [-(2n+1) + |R|], \tag{27}$$

and

$$h_{0r} = \frac{(2m)^{\frac{1}{2}}}{r_0} \exp\left[-\frac{1 \pm \mathrm{i}}{2}\left|\frac{mq_2}{2}\right|^{\frac{1}{2}} x^2\right] H_n[(\mathrm{i}mq_2)^{\frac{1}{4}}x],$$

$$h_{0\phi} = -(\mp 1 + \mathrm{i}) \left|\frac{G_0}{2}\right|^{\frac{1}{2}} \exp\left[-\frac{1 \pm \mathrm{i}}{2}\left|\frac{mq_2}{2}\right|^{\frac{1}{2}} x^2\right] H_n[(\mathrm{i}mq_2)^{\frac{1}{4}}x],$$

where the upper sign corresponds to $G_0 < 0$ and $q_2 > 0$, while the lower one corresponds to $G_0 > 0$ and $q_2 < 0$ and the signs of $h_{0\phi}$ and R can be reversed again.

The procedure of solution of the first-order equations (24), which have the following operator form

$$\hat{\mathrm{M}}_0 \boldsymbol{h}_1 = \hat{\mathrm{M}}_1(\gamma_1)\boldsymbol{h}_0,$$

is quite similar to that for the corresponding equations in the WKB method (§3). Since, owing to (23), the operator $\hat{\mathrm{M}}_0$ is degenerate, the vector of the right-hand sides of (24), $\hat{\mathrm{M}}_1(\gamma_1)\boldsymbol{h}_0$, must be orthogonal to the eigenvector $\boldsymbol{h}_0^*$ of the adjoint operator $\hat{\mathrm{M}}_0^*$ in order for the first-order equations to have non-trivial solutions. The operator $\hat{\mathrm{M}}_1(\gamma_1)$ includes only multiplications by odd powers of x and single derivatives in x. Therefore, the integrals that enter the scalar product,

$$(\boldsymbol{h}_0^* \cdot \hat{\mathrm{M}}_1 \boldsymbol{h}_0) \equiv \int_{-\infty}^{\infty} \mathrm{d}x\, \bar{h}_{0i}^* (\hat{\mathrm{M}}_1 \boldsymbol{h}_0)_i,$$

with the bar denoting the complex conjugate, vanish identically and we have $\gamma_1 = 0$. The next approximation for the eigenvalue is evaluated by Sokoloff *et al* (1989). We do not give here the rather cumbersome formulae for the next approximation, although use them in the next section.

5. Discussion

Let us illustrate our results taking as examples arbitrarily chosen simple flows with dimensionless velocity fields given by

$$\Omega = V_z = r^2 \exp(-r^2), \tag{28}$$

where the WKB asymptotics arise and

$$\Omega = \exp(-r^2), \quad V_z = (1+r^2)\exp(-r^2), \tag{29}$$

for which a magnetic layer forms.

We begin with the flow (28). It follows from (8) that

$$m/k = -1$$

and the magnetic field concentrates at the radius (see (13))

$$r_0 = \sqrt{2}.$$

Now (14) gives γ_0, while (15) or (16) and (18) determine the eigenfunction. Higher approximations for the eigenvalue are given by (19) and (21). Solid lines in figure 1 show the dependence of the field growth rate, $\mathrm{Re}\,\gamma$, on the magnetic Reynolds number, with the three lowest approximations taken into account. Notable is a very low value of the critical magnetic Reynolds number, $R_{\mathrm{m\,cr}} \approx 3$, which is even below the threshold (≈ 17) for solid-body screw motion. We should emphasize that this estimate is very uncertain and gives only the order of magnitude. Nevertheless, the flow considered may indeed be a very efficient dynamo. However, it seems to be very difficult to produce such flow in laboratory devices or meet it in nature because it does not obey the Navier–Stokes equations with any simple pressure distribution.

It goes beyond the scope of this paper to find flows that have minimal possible critical magnetic Reynolds numbers. Here we simply illustrate the difference in efficiency between different generation regimes. Consider the flow (29) for which the magnetic layer arises at the same radius, $r_0 = \sqrt{2}$, as for the flow (28). The ratio m/k can be found to satisfy

$$\frac{m}{k} = -\frac{V_z(r_0)}{\Omega(r_0)}$$

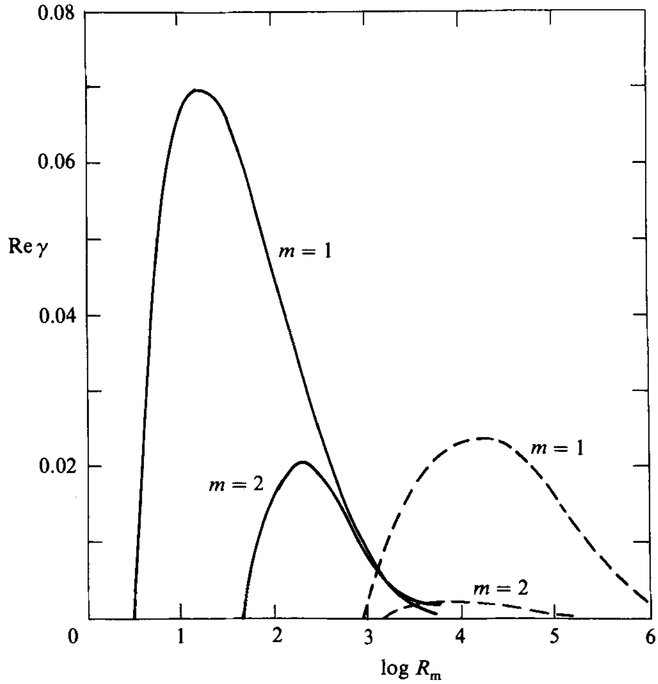

FIGURE 1. Dependence of the field growth rate of the modes $m = 1$ and $m = 2$ with $n = 0$, calculated at three leading orders in R_m, on the magnetic Reynolds number for the velocity fields (28) (WKB asymptotics, solid lines) and (29) (magnetic-layer asymptotics, dashed lines). In both flows the field concentrates at the radius $r_0 = \sqrt{2}$.

for $\operatorname{Im}\gamma = O(R_m^{-\frac{1}{2}})$. Now estimation of the eigenvalue using (27) and γ_2 calculated by Sokoloff *et al.* (1989) gives the dependence of $\operatorname{Re}\gamma$ on R_m shown in figure 1 by dashed lines. The striking difference between the generation efficiency by these two outwardly similar flows needs no comment. We see that the maximal attainable growth rate and the critical magnetic Reynolds number are very sensitive to the flow configuration.

It does not follow, however, that flows with non-identical profiles of Ω and V_z are generally less effective dynamos than those with identical profiles. One should analyse the dependence of the growth rate on the position of the magnetic layer (or, which is the same, on the longitudinal wavenumber k for fixed m). Maximization of $\operatorname{Re}\gamma_0$ from (26) or (27) gives the equation

$$(\Omega'' r - \Omega')^2 \frac{\mathrm{d}^2}{\mathrm{d}r^2}\left(\frac{\Omega V_z' - \Omega' V_z}{V_z'}\right) = -\Omega' r^3 \left[\frac{\mathrm{d}^2}{\mathrm{d}r^2}\left(\frac{\Omega V_z' - \Omega' V_z}{V_z'}\right)\right]^2$$

for the position corresponding to the maximal possible growth rate. We stress that this equation can be solved for a prescribed velocity field before calculation of the eigenvalue.

We should note that the results obtained in §3 can also be obtained by the more general boundary-layer analysis of §4. For instance, (19) can be derived by using the expansions

$$\gamma = R_m^{-\frac{1}{2}}(\gamma_0 + R_m^{-\frac{1}{2}}\gamma_1 + \dots),$$

$$r^{-1}H_\phi = \gamma_0[h_0(r) + R_m^{-\frac{1}{4}}h_{1\phi}(r) + \dots],$$

$$rH_r = -2\mathrm{i}mR_m^{-\frac{1}{2}}[h_0(r) + R_m^{-\frac{1}{4}}h_{1r}(r) + \dots],$$

and seeking a solution in the neighbourhood of the point $r = r_0$ at which (13) holds and where γ_0 is given by (14). The introduction of h_0 is motivated by the solution of the zeroth-order problem in §3. When $r_0 \neq 0$, the next-order equations follow from (4) and (5) as

$$\gamma_0(-h_{1r}+h_{1\phi}) = \gamma_1 h_0 - R_{\mathrm{m}}^{-\frac{1}{4}} h_0'',$$

$$\gamma_0(h_{1\phi}-h_{1r}) = -\gamma_1 h_0 + R_{\mathrm{m}}^{-\frac{1}{4}} h_0'' - (\mathrm{i}m/\gamma_0)(\Omega'/r)_0'' R_{\mathrm{m}}^{\frac{1}{4}}(r-r_0)^2 h_0.$$

Subtraction yields the equation

$$\frac{\mathrm{d}^2 h_0}{\mathrm{d}x^2} - (\lambda + x^2)\, h_0 = 0,$$

where
$$r - r_0 = x R_{\mathrm{m}}^{-\frac{1}{8}} \left[\frac{\mathrm{i}m}{2\gamma_0}\left(\frac{\Omega'}{r}\right)_0''\right]^{\frac{1}{4}}, \quad \gamma_1 = \lambda \left[\frac{\mathrm{i}m}{2\gamma_0}\left(\frac{\Omega'}{r}\right)_0''\right]^{\frac{1}{2}}.$$

This gives (19). We are grateful to the referee who suggested this derivation. Notice the factor $R_{\mathrm{m}}^{-\frac{1}{8}}$ in the scaling relation for $r - r_0$. It would seem that the characteristic scale of the WKB solution is $R_m^{-\frac{1}{8}}$ instead of $R_{\mathrm{m}}^{-\frac{1}{4}}$ as asserted in §3. This apparent contradiction is due to different understandings of the characteristic scale: $R_{\mathrm{m}}^{-\frac{1}{8}}$ is the scale at which the field decreases, while $R_{\mathrm{m}}^{-\frac{1}{4}}$ is the scale that determines values of radial derivatives, $\mathrm{d}H/\mathrm{d}r = O(R_{\mathrm{m}}^{\frac{1}{4}})$.

An advantage of the WKB method is that in those cases when it is applicable it always leads to an explicit solution in any order because, apart from the nonlinear Hamilton–Jacobi equation, equations of all orders are algebraic or linear first-order differential equations. The Hamilton–Jacobi equation can always be solved, at least by coordinate series expansions. On the other hand, the boundary-layer analysis leads to second-order differential equations in every order (for the induction equation), which only rarely can be solved in elementary or even special functions. Coordinate series expansions cannot be employed in the solution of boundary-layer equations since the latter are derived for the fast coordinates. A disadvantage of the WKB method is in its applicability only to equations of the special form.

An example of a helical velocity field that obeys the Navier–Stokes equations is the Couette flow between rotating concentric cylinders when the inner one moves in the axial direction. When the outer cylinder is at rest while the inner one rotates at the angular velocity Ω_1 and moves along the axis with velocity v_z, the velocity field of the fluid is given by

$$\Omega = -1 + r^{-2}, \quad V_z = -\ln r,$$

where the outer cylinder radius b is chosen as the unit length, Ω is normalized by $a^2\Omega_1/(b^2-a^2)$ and V_z normalized by $v_z/\ln(a/b)$ with a being the inner cylinder radius. The magnetic fields that concentrate at the radius r_0, with $a/b < r_0 < 1$, have the longitudinal wavenumber

$$k = -\frac{2}{m r_0^2}.$$

The main part of the oscillation frequency is given by

$$-(m\Omega_0 + kV_{z0}) = m(1-r_0^{-2}) - 2\ln r_0/(m r_0^2)$$

and the eigenvalue has the following form:

$$\gamma = \mathrm{i}\left[m\left(1-\frac{1}{r_0^2}\right) - \frac{2}{m r_0^2}\ln r_0\right] + R_{\mathrm{m}}^{-\frac{1}{2}}\left[(1+\mathrm{i})\frac{(2m)^{\frac{1}{2}}}{r_0^2}(\sqrt{2}-1-2n)\right] + O(R_{\mathrm{m}}^{-1})$$

(see (27)). Note that in this flow $\mathrm{Re}\,\gamma_0$ has the maximum exactly when the field concentrates at the inner cylinder. This introduces a correction of the order of $R_m^{-\frac{3}{4}}$ to γ because the eigenfunction can concentrate no closer than at the distance $r_0 - a = O(R_m^{-\frac{1}{4}})$ from the boundary.

These results are compatible with results of Solovyev (1985, 1987) who studied numerically the screw dynamo for the Couette–Poiseuille flow. In particular, his results show that the growth rate, $\mathrm{Re}\,\gamma$, is indeed proportional to $R_m^{-\frac{1}{2}}$.

It is interesting to compare the forms of the eigenvalue in the case of the WKB solutions, (14), (19) and (20), with that for magnetic layers, (25). The essential difference is that splitting of magnetic modes in the radial wavenumber, n, occurs in the lowest, $R_m^{-\frac{1}{2}}$, order for the magnetic layers, while the splitting is much weaker, of the order $R_m^{-\frac{3}{4}}$, for the WKB asymptotics.

Finally, we note that in realistic flows, e.g. in astrophysical jets, the hydrodynamic Reynolds numbers are very high and the flow is turbulent. In this case the solutions obtained here describe the behaviour of the mean magnetic field and ν_m should be replaced by the sum of the turbulent and Ohmic diffusivities. In astrophysical jets the turbulent magnetic Reynolds numbers can be rather large. Indeed, estimation of the turbulent diffusivity by $\nu_t \approx l v_s/3$, where $l \approx \frac{1}{10} r_*$ is the turbulent scale and v_s is the sound speed, yields $R_m \approx 30M$, where $M = V_z/v_s$ is the Mach number, which is of order 1–10 in the jets (see Bridle & Perley 1984).

One more class of objects where the screw dynamo may be operative are rotating planetary liquid cores where the fluid flow may acquire a helical pattern due to meridional circulation.

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